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Lindblad dynamics of Gaussian states and their superpositions in the semiclassical limit

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Abstract. The time evolution of the Wigner function for Gaussian states generated by Lindblad quantum dynamics is investigated in the semiclassical limit. A new type of classical phase-space dynamics is obtained, where the Lindblad terms generally introduce a non-Hamiltonian flow. In addition to this classical phase-space dynamics, the Gaussian approximation yields dynamical equations for the covariances. The approximation becomes exact for linear Lindblad operators and a quadratic Hamiltonian. By viewing the Wigner function as a wave function on a coordinate space of doubled dimension, and the phase-space Lindblad equation as a Schrödinger equation with a non-Hermitian Hamiltonian, a further set of semiclassical equations are derived. These are also capable of describing the interference terms in Wigner functions arising in superpositions of Gaussian states, as demonstrated for a cat state in an anharmonic oscillator subject to damping.

1. Introduction

Since the early days of quantum mechanics the dynamics of Gaussian wave packets has been considered as a natural connection between quantum and classical dynamics [1,2]. As shown by Heller, Hepp and Littlejohn [3–5], in closed quantum systems the motion of the centre in the semiclassical limit is described by Hamilton’s equations of motion. Semiclassical methods based on the time evolution of Gaussian states along classical trajectories provide powerful numerical and analytical tools [6–11].

Realistic quantum systems, however, are open. That is, they exchange energy with their environment. In the Markovian approximation a system weakly coupled to its environment can be described by a Lindblad equation (see, e.g. [12]). Markovian open quantum systems play a crucial role in various branches of quantum physics ranging from quantum optics and information to atomic, nuclear, and condensed matter physics. It is an interesting question how the dynamics of Gaussian wave packets generalise in this context, and what can be learned from the semiclassical limit. Here we generalise the approach of Heller and Littlejohn [3, 5] to Lindblad type quantum dynamics. This yields a new type of classical phase-space dynamics, where the Lindblad terms generally introduce non-Hamiltonian flows, which in special cases can take the form of a gradient flow. Furthermore, the semiclassical dynamics yields an approximation of the quantum covariances, which can be useful in various applications.

The current study complements previous investigations of the semiclassical limit of Lindblad dynamics, using the framework of path integrals [13], and complex WKB dynamics [14]. In [15] first steps have been made towards applying Heller's wave packet method to Lindblad dynamics for the case of linear Lindblad operators. There the framework of chord functions in a doubled phase-space is used. This differs from the approach taken here, which is a more direct application of Heller's method.

The paper is organised as follows. We first introduce the Lindblad equation on phase-space in section 2 and make a Gaussian ansatz for the state, from which semiclassical equations of motion for the centre and covariance matrix are derived in section 3. The resulting semiclassical equations are interpreted and the conditions under which the centre dynamics may be written as a gradient flow are discussed. We transform our equations of motion to a form that is better suited to many applications, particularly those in quantum optics. The example of an oscillator with nonlinear losses and amplification is used to illustrate this. Finally, by viewing the phase-space Lindblad equation as a Schrödinger equation with a non-Hermitian Hamiltonian, we connect the Lindblad dynamics to non-Hermitian quantum dynamics in section 4. The semiclassical limit of the latter has been investigated in detail by two of the authors in [16, 17]. Applying results from this context allows us to derive a further set of semiclassical equations for Lindblad dynamics. These are capable of describing the interference terms in Wigner functions arising, for example, in superpositions of Gaussian states. We demonstrate this for a cat state in a damped anharmonic oscillator. We conclude with a short summary.

2. Lindblad Equation on Phase-Space

We consider the dynamics of quantum systems generated by equations of Lindblad type

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}] + i \sum_k \hat{L}_k \hat{\rho} \hat{L}_k^\dagger - \frac{1}{2} \hat{L}_k^\dagger \hat{L}_k \hat{\rho} - \frac{1}{2} \hat{\rho} \hat{L}_k^\dagger \hat{L}_k, \quad (1)$$

where $\hat{\rho}$ is the density matrix describing the state of the quantum system. The first term corresponds to the unitary dynamics generated by the Hamiltonian \hat{H} , while the following terms containing the Lindblad operators \hat{L}_k account for (weak) interactions of the system with an environment. For more details see, e.g., [12].

The phase-space representation of quantum mechanics [18] is particularly convenient for analysing the semiclassical limit. In the Wigner-Weyl representation an operator \hat{f} on Hilbert space is mapped to a phase-space function $f(x)$, known as the Weyl symbol of \hat{f} , via the Wigner-Weyl transformation

$$f(x) = \int d\xi \langle q + \frac{\xi}{2} | \hat{f} | q - \frac{\xi}{2} \rangle e^{-ip \cdot \xi / \hbar}, \quad (2)$$

where the canonical coordinates $x = (q, p)$ span the $2n$ -dimensional phase-space. In this representation the quantum state $\hat{\rho}$ is represented by the Wigner function, which is defined by the Wigner-Weyl transformation of the operator $\hat{\rho}$ as

$$W(x) = \frac{1}{(2\pi\hbar)^n} \int d\xi \langle q + \frac{\xi}{2} | \hat{\rho} | q - \frac{\xi}{2} \rangle e^{-p \cdot \xi / \hbar}. \quad (3)$$

Assuming that the Hamiltonian \hat{H} and the Lindblad operators \hat{L}_k are the Weyl quantisations of sufficiently well-behaved phase-space functions $H(x)$ and $L_k(x)$, the

evolution equation for the Wigner function is given by

$$i\hbar \frac{\partial W}{\partial t} = (H \star W - W \star H) + i \sum_k L_k \star (W \star \bar{L}_k) - \frac{1}{2} \bar{L}_k \star (L_k \star W) - \frac{1}{2} W \star (\bar{L}_k \star L_k), \quad (4)$$

where $f \star g$ denotes the Moyal product of two phase-space functions f and g ,

$$(f \star g)(x) = f(x) e^{\frac{i\hbar}{2} \overleftarrow{\nabla} \cdot \Omega \overrightarrow{\nabla}} g(x), \quad (5)$$

$$= f(x)g(x) + \frac{i\hbar}{2} \{f(x), g(x)\} + \dots \quad (6)$$

Here Ω denotes the symplectic form

$$\Omega = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix}, \quad (7)$$

the phase-space gradient is $\nabla := (\partial_q, \partial_p)$ and the arrows over the differential operators in (5) indicate whether they act on the function to the left or to the right. The first two terms of the Moyal product are shown in (6), where $\{A, B\} = \nabla A \cdot \Omega \nabla B$ is the usual Poisson bracket.

Following the work of Heller, Hepp and Littlejohn [3–5] for closed systems, assuming that the initial state is a well-localised Gaussian state, it is justified to approximate the Hamiltonian and the Lindblad operators by finite Taylor expansions around the centre of the state. In particular, expanding the Hamiltonian up to second order and the Lindblad operators up to first order leads to dynamics that preserve the Gaussian nature of the state. We can therefore make the ansatz that the time-evolved state remains Gaussian for all times and instantaneously expand the Hamiltonian and the Lindblad operators in Taylor series around the time-dependent centre of the state. For closed quantum systems this approximation yields Hamilton's canonical equations of motion for the centre of the state, while the width changes with the linearised classical flow. In what follows we shall investigate how this dynamics generalises in the presence of Lindblad operators.

3. Gaussian Evolution in the Semiclassical Limit

3.1. Deriving semiclassical equations of motion

The Wigner function of a general Gaussian state is of the form

$$W(x) = \frac{\sqrt{\det G}}{(\pi\hbar)^n} e^{-(x-X) \cdot G(x-X)/\hbar}, \quad (8)$$

where $x = (q, p) \in \mathbb{R}^n \times \mathbb{R}^n$ are canonical phase-space coordinates, X is a vector of the expectation values of the quantum position and momentum operators

$$X_k = \langle \hat{x}_k \rangle \equiv \text{Tr}(\hat{\rho} \hat{x}_k), \quad (9)$$

with $\hat{x} = (\hat{q}_1, \dots, \hat{q}_n, \hat{p}_1, \dots, \hat{p}_n)$, and G is a real, symmetric and positive definite matrix, encoding the width of the wave packet via the (co)variances of the canonical operators as

$$\hbar (G^{-1})_{jk} = \langle \hat{x}_j \hat{x}_k + \hat{x}_k \hat{x}_j \rangle - 2\langle \hat{x}_j \rangle \langle \hat{x}_k \rangle. \quad (10)$$

An initial Gaussian state, with time-dependent parameters G and X_k , remains Gaussian for all times under the quantum dynamics generated by (4) if the Hamiltonian is at most quadratic and the Lindblad operators are linear in \hat{x} . As a Gaussian Wigner function of the form (8) is localised in a small area around its centre, we can Taylor expand the Hamiltonian and the Lindblad operators to second and first order around this centre. In the neighbourhood of the centre the higher derivatives of the Hamiltonian and the Lindblad operators are negligible, while further away from the centre, where higher orders are important, the Wigner function is negligible. Thus, by making use of (5), the evolution equation for the Wigner function (4) simplifies to the semiclassical form

$$\begin{aligned} i\hbar \frac{\partial W}{\partial t} = i\hbar \{H, W\} + \frac{\hbar}{2} \sum_k \{\bar{L}_k, L_k W\} - \{L_k, \bar{L}_k W\} \\ + \frac{i\hbar^2}{4} \sum_k \{L_k, \{\bar{L}_k, W\}\} + \{\bar{L}_k, \{L_k, W\}\} + \dots, \end{aligned} \quad (11)$$

where H and L_k are the second and first order Taylor expansions of the Weyl symbols of the Hamiltonian and the Lindblad operators around the centre X

$$H(x) \approx H(X) + \nabla H(X) \cdot \delta x + \frac{1}{2} \delta x \cdot H''(X) \delta x, \quad (12)$$

$$L_k(x) \approx L_k(X) + \nabla L_k(X) \cdot \delta x, \quad (13)$$

with $\delta x = x - X$. Inserting (12), (13) and the Gaussian ansatz (8) with time-dependent parameters into (11) yields

$$\begin{aligned} \left[\frac{i\hbar}{2} \text{Tr}(G^{-1} \dot{G}) + 2iG\dot{X} \cdot \delta x - i\dot{G}\delta x \cdot \delta x \right] W = \left[i\hbar \sum_k \nabla L_k(X) \cdot \Omega G \Omega \nabla \bar{L}_k(X) \right. \\ - \hbar \sum_k \nabla L_k(X) \cdot \Omega \nabla \bar{L}_k(X) + 2iG\Omega \nabla H(X) \cdot \delta x + 2iG\Omega \sum_k \text{Im}(L_k(X) \nabla \bar{L}_k(X)) \cdot \delta x \\ - i(H''\Omega G - G\Omega H'')\delta x \cdot \delta x + 2i \sum_k (\nabla L_k(X) \cdot \Omega G \delta x) (\nabla \bar{L}_k(X) \cdot \Omega G \delta x) \\ \left. + \sum_k (\nabla \bar{L}_k(X) \cdot \delta x) (\nabla L_k(X) \cdot \Omega G \delta x) - (\nabla L_k(X) \cdot \delta x) (\nabla \bar{L}_k(X) \cdot \Omega G \delta x) \right] W. \end{aligned} \quad (14)$$

Separating different powers of δx then leads to the semiclassical equations of motion for the parameters X and G

$$\dot{X} = \Omega \nabla H + \Omega \sum_k \text{Im}(L_k \nabla \bar{L}_k), \quad (15)$$

$$\dot{G} = H''\Omega G - G\Omega H'' + (\Lambda \Omega G + G\Omega \Lambda) + 2G\Omega D \Omega G. \quad (16)$$

Here we have defined

$$\Lambda = \sum_k \text{Im}(\nabla L_k \nabla \bar{L}_k^T), \quad (17)$$

$$D = \sum_k \text{Re}(\nabla L_k \nabla \bar{L}_k^T), \quad (18)$$

H'' is the Hessian matrix and all phase-space functions are evaluated at the centre of the Gaussian, X . To obtain (16) the symmetry enforcing convention $G = (G + G^T)/2$ was applied. As the Wigner function depends only on the symmetric part of G , any antisymmetric part is unobservable.

Equations (15) and (16) are two of the main results of the present paper. To the best of our knowledge they constitute a new type of classical phase-space dynamics. As expected, in the unitary case the centre moves according to the classical canonical equations of motion $\dot{X} = \Omega \nabla H$ and the evolution of G is governed by the linearised Hamiltonian flow around the classical trajectory. The general dynamical equations for the centre of the state can be interpreted as a generalisation of Hamilton's canonical equations, arising as the classical counterpart of quantum Lindblad evolution. Following Littlejohn [5], the terms $H''\Omega G - G\Omega H'' + (\Lambda\Omega G + G\Omega\Lambda)$ can be interpreted as a linearised flow in the non-unitary case. However, the additional term $2G\Omega D\Omega G$ in (16) does not result from the linearised flow. This term originates from the double Poisson brackets in (11) and ensures that the Heisenberg uncertainty principle is not violated. In particular, a physically meaningful Gaussian state must fulfil the Robertson-Schrödinger uncertainty relation, expressed in terms of G as [19]

$$G^{-1} + i\Omega \geq 0. \quad (19)$$

The extra term $2G\Omega D\Omega G$ is a quantum correction appearing in the semiclassical dynamics, guaranteeing that (19) is fulfilled for all times.

While the generalisation of Heller's theory to Markovian open systems with *linear* Lindblad operators has previously been considered in [14, 15], the approach taken therein is quite different from the one presented here, and no equivalent of equations (15) and (16) is derived. In particular, the authors work with the Fourier transform of the Wigner function, the so-called chord function, in double phase-space. Of course, both approaches are exact for quadratic Hamiltonians and linear Lindblad operators. For more general systems they yield different approximations (resulting from a local expansion of the Hamiltonian and Lindblad terms in real space or Fourier space). The present approach has the advantage of yielding the set of dynamical equations (15) and (16) for the position and momentum expectation values as well as the covariances, which can be interpreted as a new type of classical dissipative phase-space dynamics.

3.2. Geometric interpretation of the Lindblad terms

We now analyse the structure of the Lindblad terms in the dynamical equation for the centre (15) in more detail. There are two cases for which the geometric interpretation of these terms is simple: First, for purely Hermitian or anti-Hermitian Lindblad operators the flow generated by the Lindblad terms vanishes, and the only effect in the semiclassical description is on the width of the Wigner function. This is in line with the well-known result that purely Hermitian or anti-Hermitian Lindblad operators lead to decoherence but no dissipation [20].

The second case is where the Weyl symbols of the Lindblad operators are analytic functions of $q \pm ip$. In this case, the flow they generate can be written as a gradient flow of the phase-space function $\Gamma := \mp \frac{1}{2}|L|^2$. In fact, the flow generated by a Lindblad operator is the gradient flow of the function $\Gamma = \mp \frac{1}{2}|L|^2$ if and only if the Lindblad symbol is a holomorphic function of either $q + ip$ or $q - ip$. This can be seen as follows. The holomorphy of L as a function of $q \pm ip$ is equivalent to the validity of

the Cauchy-Riemann conditions

$$\nabla \operatorname{Re}(L) = \pm \Omega \nabla \operatorname{Im}(L), \quad (20)$$

which imply

$$\nabla \operatorname{Im}(L) = \mp \Omega \nabla \operatorname{Re}(L). \quad (21)$$

Observing that the Lindblad term on the right hand side of (15) may be expressed as

$$\Omega \operatorname{Im}(L \nabla \bar{L}) = \operatorname{Im}(L) \Omega \nabla \operatorname{Re}(L) - \operatorname{Re}(L) \Omega \nabla \operatorname{Im}(L), \quad (22)$$

the right hand side can immediately be identified as

$$\mp (\operatorname{Re}(L) \nabla \operatorname{Re}(L) + \operatorname{Im}(L) \nabla \operatorname{Im}(L)) = \mp \frac{1}{2} \nabla |L|^2. \quad (23)$$

On the other hand, assume we have

$$\operatorname{Im}(L) \Omega \nabla \operatorname{Re}(L) - \operatorname{Re}(L) \Omega \nabla \operatorname{Im}(L) = \mp \frac{1}{2} \nabla |L|^2 = \mp (\operatorname{Re}(L) \nabla \operatorname{Re}(L) + \operatorname{Im}(L) \nabla \operatorname{Im}(L)) \quad (24)$$

and that $\operatorname{Re}(L) \neq 0$ and $\operatorname{Im}(L) \neq 0$ (as otherwise the Lindblad flow vanishes). Acting with the symplectic onto equation (24) gives

$$- \operatorname{Im}(L) \nabla \operatorname{Re}(L) + \operatorname{Re}(L) \nabla \operatorname{Im}(L) = \mp (\operatorname{Re}(L) \Omega \nabla \operatorname{Re}(L) + \operatorname{Im}(L) \Omega \nabla \operatorname{Im}(L)). \quad (25)$$

Combining this with the original expression (24) yields the Cauchy-Riemann conditions $\nabla \operatorname{Re}(L) = \pm \Omega \nabla \operatorname{Im}(L)$. The gradient dynamics drives trajectories towards the closest maximum of the function Γ . Thus one possible physical interpretation is that Γ is an entropy of the system. Note that similar gradient dynamics have been discussed in the context of thermodynamics e.g. by Öttinger and Grmela [21, 22].

Of course, there are operators that are neither Hermitian or anti-Hermitian nor have symbols that are analytic functions of $q \pm ip$. In some of these cases the Lindblad terms can still be written as gradient flows of more general functions. For instance, Lindblad operators of the form $\hat{L} = a\hat{q} + ib\hat{p}$ with $a \neq \pm b$ result in a semiclassical flow given by the gradient of the function $\Gamma = -\frac{ab}{2}(q^2 + p^2) \neq \mp \frac{1}{2}|L|^2$.

There are other Lindblad operators that instead lead to Hamiltonian flows, such as normal operators in one dimension. However, in general the Lindblad term leads to a flow which is neither a Hamiltonian nor a gradient flow. Instead of deriving more intricate mathematical conditions for different types of flows, the following two examples aim to develop a better intuition of Lindblad operators that do not generate a Hamiltonian or gradient flow.

Example 1: Consider a two-dimensional system and a linear Lindblad operator $\hat{L} = \sqrt{\gamma}(\hat{q}_1 + i\hat{p}_2)$. Without a Hamiltonian term the evolution equations for the centre are

$$\dot{q}_1 = 0, \quad \dot{q}_2 = -\gamma q_1, \quad \dot{p}_1 = -\gamma p_2, \quad \dot{p}_2 = 0. \quad (26)$$

In this case the semiclassical approximation is exact since L is linear. The flow is neither Hamiltonian nor a gradient flow. Both q_1 and p_2 are constants of motion and $p_1(q_2) = \frac{p_2}{q_1} q_2 + \frac{p_1(0)}{q_2(0)}$, i.e. all straight lines are phase-space trajectories and all points in the plane $(0, q_2, p_1, 0)$ in the four-dimensional phase-space are fixed points.

Example 2: Consider a one-dimensional system with nonlinear Lindblad operator $\hat{L} = \sqrt{\gamma}(\hat{q}^2 + i\hat{p}^2)$. The semiclassical equations of motion for q, p are found to be

$$\begin{aligned}\dot{q} &= -2\gamma q^2 p, \\ \dot{p} &= -2\gamma q p^2,\end{aligned}\tag{27}$$

which again is clearly neither a pure Hamiltonian nor a pure gradient flow. The phase-space portrait of this dynamics is depicted in Figure 1. The flow conserves the quantity p/q and thus the trajectories are straight lines. The lines $q = 0$ and $p = 0$ are fixed points. In particular the point $p = 0 = q$ acts as a hyperbolic fixed point. Furthermore, the lines $q = p$ and $q = -p$ are the stable and unstable manifolds, respectively.

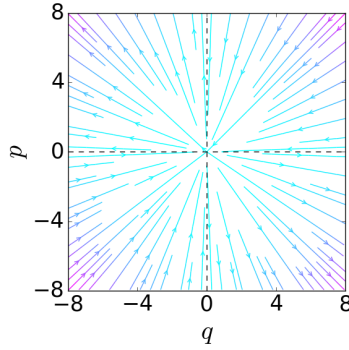


Figure 1: The phase-space portrait for the classical dynamics (27) with $\gamma = 0.1$. False colours indicate the velocity.

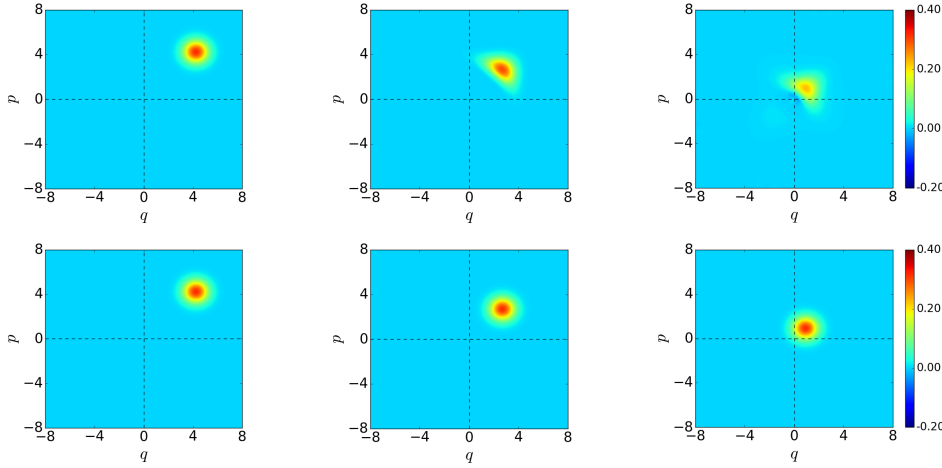


Figure 2: Comparison of the quantum (top row) and semiclassical (bottom row) dynamics of an initial Glauber coherent state starting on the stable manifold at $(q, p) = (3\sqrt{2}, 3\sqrt{2})$ with $\gamma = 0.1$. Times $t = 0, 0.2, 2.8$ are shown left to right.

In this example the semiclassical description is of course only an approximation. Let us briefly discuss the quantum-classical correspondence. For this purpose we depict

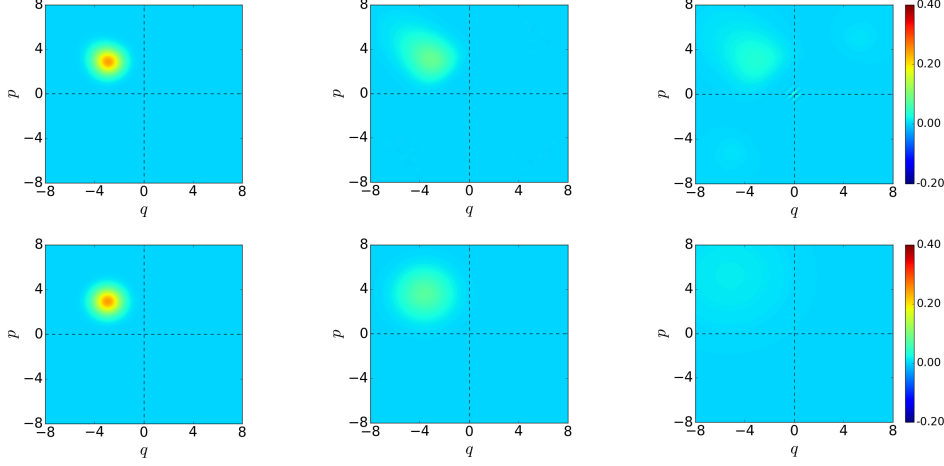


Figure 3: Comparison of the quantum (top row) and semiclassical (bottom row) dynamics of an initial Glauber coherent state starting on the unstable manifold at $(q, p) = (-2\sqrt{2}, 2\sqrt{2})$ with $\gamma = 0.1$. Times $t = 0.02, 0.12, 0.22$ are shown left to right.

the Wigner function obtained from the numerically exact propagation in comparison to the Gaussian approximation for two selected initial states in figures 2 and 3. For a Glauber coherent state initially centred on the stable manifold (Figure 2) the quantum Wigner function deforms in a manner consistent with the classical flow (Figure 1) and makes a very slow approach to the centre, as anticipated from the velocity of the flow. As the initial Wigner function is a minimum uncertainty state, and the semiclassical Wigner function must remain Gaussian, its shape cannot change and thus remains a Glauber coherent state for all time.

The Wigner function of a Glauber coherent state initially centred on the unstable manifold quickly delocalises (Figure 3). In the quantum dynamics two dense regions begin to emerge where the spreading distribution appears to be 'caught' in the two stable quadrants, resulting in an interference pattern at the origin. The semiclassical centre follows the flow illustrated in Figure 1 and for short times provides a reasonable approximation of the quantum dynamics. However, as expected, the approximation clearly breaks down at longer times.

3.3. Formulation in creation and annihilation operators

In many applications, in particular in quantum optics, the Hamiltonian and Lindblad operators are often expressed in terms of annihilation and creation operators \hat{a}_j and \hat{a}_j^\dagger , satisfying the commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \quad [\hat{a}_i, \hat{a}_j] = 0. \quad (28)$$

For such systems it is convenient to express the semiclassical equations (15) and (16) in terms of the complex canonical phase-space variables

$$a_j = \frac{1}{\sqrt{2}}(q_j + ip_j), \quad (29)$$

where q_j and p_j are the classical counterparts of the quadrature operators, defined as

$$\hat{q}_j = \frac{1}{\sqrt{2}}(\hat{a}_j^\dagger + \hat{a}_j), \quad \hat{p}_j = \frac{i}{\sqrt{2}}(\hat{a}_j^\dagger - \hat{a}_j), \quad (30)$$

which are in general not associated to a physical position or momentum. For the remainder of this subsection we choose to work in units of $\hbar = 1$ for simplicity.

The operators \hat{q}_j, \hat{p}_j can be grouped into the vector $\hat{x} = (\hat{q}_1, \dots, \hat{q}_n, \hat{p}_1, \dots, \hat{p}_n)$ and the commutation relations between the quadrature operators can then be written in the form

$$[\hat{x}_i, \hat{x}_j] = i\Omega_{ij}, \quad (31)$$

where Ω is the symplectic form (7). That is, they form a set of canonically conjugate observables. If the Hamiltonian and Lindblad operators are expressed in terms of the \hat{x}_j , then the semiclassical Gaussian dynamics derived in the previous section are applicable.

The transformation from quadrature operators \hat{q}, \hat{p} to mode operators \hat{a}, \hat{a}^\dagger is achieved via the transformation matrix [19]

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix}. \quad (32)$$

Applying the transformations $\dot{A} = T\dot{X}$ and $\dot{M} = T\dot{G}T^\dagger$ yields

$$\dot{A} = -i\Omega\nabla H - \frac{1}{2}\Omega \sum_k (L_k \nabla \bar{L}_k - \bar{L}_k \nabla L_k), \quad (33)$$

$$\dot{M} = i(M\Omega H'' + \bar{H}''\Omega M) - (M\Omega\Gamma + \bar{\Gamma}\Omega M) + 2M\Omega\Xi\Omega M, \quad (34)$$

where $A = (a, \bar{a})$, $\nabla := (\partial_a, \partial_{\bar{a}})$ is the gradient operator and H'' is the Hessian. We have also defined Γ and Ξ as

$$\Gamma = \frac{1}{2} \sum_k (\nabla L_k \nabla \bar{L}_k^\top - \nabla \bar{L}_k \nabla L_k^\top), \quad (35)$$

$$\Xi = \frac{1}{2} \sum_k (\nabla L_k \nabla \bar{L}_k^\top + \nabla \bar{L}_k \nabla L_k^\top) \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}. \quad (36)$$

In the complex coordinates (a, \bar{a}) the covariance matrix $\Sigma = M^{-1}$ takes the block form

$$\Sigma = \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \beta & \alpha \end{pmatrix}, \quad (37)$$

where

$$\alpha_{ij} = \langle \hat{a}_i^\dagger \hat{a}_j + \hat{a}_j \hat{a}_i^\dagger \rangle - 2\langle \hat{a}_i^\dagger \rangle \langle \hat{a}_j \rangle, \quad (38)$$

$$\beta_{ij} = \langle \hat{a}_i^\dagger \hat{a}_j^\dagger + \hat{a}_j^\dagger \hat{a}_i^\dagger \rangle - 2\langle \hat{a}_i^\dagger \rangle \langle \hat{a}_j^\dagger \rangle. \quad (39)$$

Example 3: Let us apply this formulation to an example of a harmonic oscillator with nonlinear damping and amplification. The Hamiltonian is given by

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} \quad (40)$$

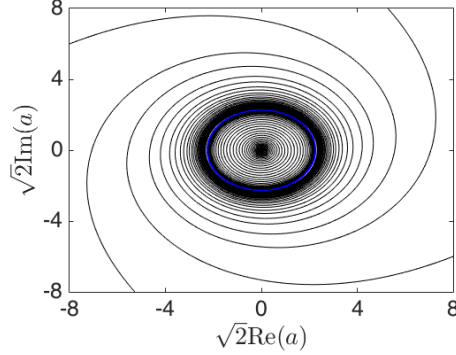


Figure 4: A phase-space portrait for the classical dynamics (42) with $\omega = 1$, $\gamma = 0.1$, $\Gamma = 0.01$ and $A = 0.15$. There is a stable limit cycle at $|a|^2 = 2.5$, indicated by the blue circle, and the origin is an unstable fixed point.

and the three Lindblad operators

$$\hat{L}_1 = \sqrt{\gamma}\hat{a}, \quad \hat{L}_2 = \sqrt{\Gamma}\hat{a}^2 \quad \text{and} \quad \hat{L}_3 = \sqrt{A}\hat{a}^\dagger \quad (41)$$

describe the damping and gain, where γ , Γ and A are the linear damping rate, nonlinear damping rate and amplification rate respectively. This model appears in the context of quantum optomechanics and is discussed in [23]. For instance, it could describe a driven nanomechanical oscillator coupled to a thermal bath, where the damping rate depends on the excitation of the resonator. Applying (33) immediately yields the semiclassical equation

$$\dot{a} = -i\omega a + \frac{1}{2}(A - \gamma)a - \Gamma|a|^2 a. \quad (42)$$

As discussed in [23] the origin $|a| = 0$ is a stable fixed point provided $A - \gamma < 0$. However, when the value of A exceeds γ the system exhibits a Hopf bifurcation and the origin becomes unstable. In this case a stable limit cycle occurs and the long-term solution tends towards the curve $|a|^2 = (A - \gamma)/2\Gamma$. A phase-space portrait for the case $A > \gamma$ is illustrated in Figure 4 and the semiclassical and quantum dynamics are compared in Figure 5. Due to the weak nonlinear damping a good correspondence is observed for short to medium times. Over longer periods of time the semiclassical Gaussian becomes trapped on the limit cycle and the width no longer changes. On the other hand, in accordance with the classical flow (Figure 4), the quantum Wigner function begins to smear out over the limit cycle into a 'donut' shape.

In Figure 6 we additionally examine the time evolution of the covariance element α , where α is defined in (38). As expected, in the short time limit there is a good agreement with the quantum dynamics. Over a longer time, once the semiclassical solution $|a|$ is close to the limit cycle, the value of α plateaus, reproducing a feature also present in quantum dynamics. This feature appears to stem from the quantum Wigner function smearing out over the limit cycle. It should be stressed, however, that in general the semiclassical approximation is only valid for short times. Indeed, even in this example one can choose initial conditions such that the *long-time* semiclassical dynamics based on a quadratic approximation around the centre are not even qualitatively correct. In order to describe the spread of the Wignerfunction over

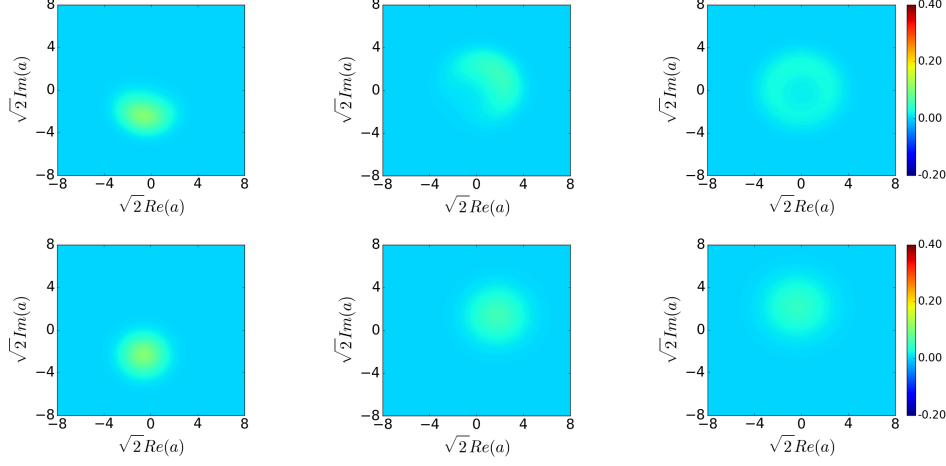


Figure 5: Comparison of the quantum (top row) and semiclassical (bottom row) dynamics of an initial Glauber coherent state with $\sqrt{2} \text{Re}(a) = 1.5$ and $\sqrt{2} \text{Im}(a) = 2$. The harmonic oscillator frequency $\omega = 1$, while the linear damping rate, nonlinear damping rate and amplification rate are $\gamma = 0.1$, $\Gamma = 0.01$ and $A = 0.15$ respectively. Times $t = 9, 38, 150$ are shown left to right.

a limit cycle in a semiclassical framework we would have to adapt the methods and results from [24] to open systems.

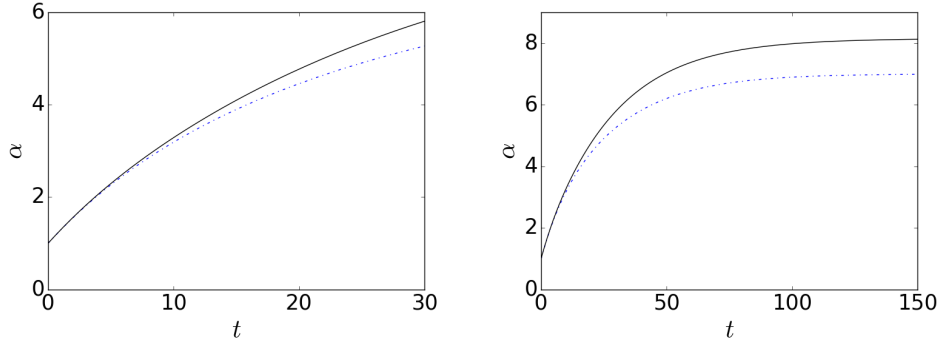


Figure 6: Time evolution of the covariance element $\alpha = \langle \hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger \rangle - 2\langle \hat{a}^\dagger \rangle \langle \hat{a} \rangle$ for short (left) and long (right) periods of time. The quantum dynamics (black) are compared to the semiclassical results (blue dashed), with the same initial conditions and parameters as Figure 5.

4. Lindblad Dynamics as Schrödinger Dynamics with a Non-Hermitian Hamiltonian

In this section we approach the Lindblad dynamics from a different perspective, specifically, by interpreting the Lindblad equation as a Schrödinger equation with a non-Hermitian Hamiltonian. The notion of a phase-space Schrödinger equation

and phase-space wave functions is not new. Notably, for closed systems, Koda recently formulated several initial value semiclassical propagators for the Wigner function starting from an interpretation of the Moyal equation as a Schrödinger equation [25]. In the case of Lindblad dynamics, the new ingredient is the non-Hermiticity of the resulting Hamiltonian operator. Quantum dynamics generated by non-Hermitian Hamiltonians has in recent years attracted considerable attention in its own right (see, e.g., [26, 27], and references therein). In [16, 17] two of the authors have developed the semiclassical limit of Gaussian wave packet propagation for non-Hermitian Hamiltonians. Reinterpreting the Wigner-evolution for Lindblad systems as a non-Hermitian Schrödinger equation, we can directly apply these results to obtain dynamical equations that can describe the semiclassical propagation of complex Gaussian Wigner functions, as they appear for example in the superpositions of Gaussian wave packets.

Let us for example consider an initial state $\phi(q) = \sum_j c_j \phi_j^A(q)$ that is a superposition of Gaussian wave packets

$$\phi_j^A(q) = \frac{(\det \operatorname{Im} A)^{1/4}}{(\pi \hbar)^{n/4}} e^{\frac{i}{\hbar} [\frac{1}{2}(q-q_j) \cdot A (q-q_j) + p_j \cdot (q-q_j)]}, \quad (43)$$

where $q, q_j, p_j \in \mathbb{R}^n$ and A is an $n \times n$ complex symmetric matrix with $\operatorname{Im} A > 0$. The Wigner function of this state is

$$W(x) = \sum_{i,j} c_i^* c_j \psi_{ij}(x), \quad (44)$$

where $x = (q, p)$ and

$$\psi_{ij}(x) = \frac{1}{(\pi \hbar)^n} e^{\frac{i}{\hbar} [(x - X_{ij}) \cdot iG(x - X_{ij}) + Y_{ij} \cdot (x - X_{ij}) + \alpha_{ij}]} \quad (45)$$

is a complex Gaussian centred at $X_{ij} = \frac{1}{2}(q_i + q_j, p_i + p_j)$ with 'momentum' $Y_{ij} = (p_j - p_i, q_i - q_j)$ and a complex phase $\alpha_{ij} = \frac{1}{2}(p_i + p_j) \cdot (q_i - q_j)$. The matrix G is related to the width A of the Gaussians in the superposition by

$$G = \begin{pmatrix} \operatorname{Im} A + \operatorname{Re} A [\operatorname{Im} A]^{-1} \operatorname{Re} A & -\operatorname{Re} A [\operatorname{Im} A]^{-1} \\ -[\operatorname{Im} A]^{-1} \operatorname{Re} A & [\operatorname{Im} A]^{-1} \end{pmatrix}. \quad (46)$$

As the phase-space Lindblad equation (4) is linear in the Wigner function W , the time evolution of (44) can be obtained by evolving each complex Gaussian ψ_{ij} individually and summing the results.

With this picture in mind our starting point is once again the Lindblad equation on phase space

$$i\hbar \frac{\partial \psi}{\partial t} = (H \star \psi - \psi \star H) + i \sum_k L_k \star (\psi \star \bar{L}_k) - \frac{1}{2} \bar{L}_k \star (L_k \star \psi) - \frac{1}{2} \psi \star (\bar{L}_k \star L_k). \quad (47)$$

However, we have now switched notation from W to ψ to indicate that we could be dealing with a complex component of the Wigner function, such as the ψ_{ij} in the discussion above. As this equation is linear in ψ we can in fact view (47) as a Schrödinger equation with a non-Hermitian Hamiltonian in a larger Hilbert space.

To this end we will use the fact that the star product can be written as the action of an operator on phase-space functions, see e.g. [25]. More precisely, we have

$$A \star \psi = \hat{A}^{(-)} \psi \quad \text{and} \quad \psi \star A = \hat{A}^{(+)} \psi, \quad (48)$$

where the operators

$$\hat{A}^{(\pm)} = A(\hat{x} \pm \frac{1}{2}\Omega\hat{y}) \quad (49)$$

act on the phase-space function $\psi(x)$. Furthermore,

$$\hat{x} = (q, p), \quad (50)$$

$$\hat{y} = (-i\hbar\nabla_q, -i\hbar\nabla_p), \quad (51)$$

are a pair of Hermitian operators that satisfy the canonical commutation relations

$$[\hat{x}_i, \hat{y}_j] = i\hbar\delta_{ij} \quad (52)$$

and can thus be treated like position and momentum operators in a space of doubled dimension. Making use of (48), the phase-space Lindblad equation (47) can be written as a Schrödinger equation

$$i\hbar \frac{\partial W(x, t)}{\partial t} = \hat{K}(\hat{x}, \hat{y})W(x, t), \quad (53)$$

with non-Hermitian Hamiltonian

$$\hat{K}(\hat{x}, \hat{y}) = \hat{H}^{(-)} - \hat{H}^{(+)} + i \sum_k \hat{L}_k^{(-)} \hat{L}_k^{(+)} - (\widehat{\bar{L}_k \star L_k})^{(-)} - (\widehat{\bar{L}_k \star L_k})^{(+)}. \quad (54)$$

Finally, through the Wigner-Weyl transformation the non-Hermitian Hamiltonian \hat{K} can be mapped onto the double phase-space function

$$\begin{aligned} K(x, y) &= H(x - \frac{1}{2}\Omega y) - H(x + \frac{1}{2}\Omega y) + i \sum_k L_k(x - \frac{1}{2}\Omega y) \star_2 \bar{L}_k(x + \frac{1}{2}\Omega y) \\ &\quad - \frac{1}{2}(\bar{L}_k \star L_k)(x - \frac{1}{2}\Omega y) - \frac{1}{2}(\bar{L}_k \star L_k)(x + \frac{1}{2}\Omega y), \end{aligned} \quad (55)$$

where \star_2 denotes the Moyal product on the double phase-space.

We are now in the position to study the time evolution of initial wave packets of the form

$$\psi(x) = \frac{(\det \text{Im } B)^{1/4}}{(\pi\hbar)^{n/2}} e^{\frac{i}{\hbar}[(x-X) \cdot B(x-X)/2 + Y \cdot (x-X) + \alpha]}, \quad (56)$$

where $X, Y \in \mathbb{R}^{2n}$, $\alpha \in \mathbb{C}$ is a phase factor and B is a complex symmetric matrix with $\text{Im } B > 0$. As shown above, such states appear as components of the Wigner function of superpositions of Gaussian states (44).

Direct application of the semiclassical equations derived in [16, 17] yields the following equations of motion

$$\dot{Z} = \Omega_2 \nabla \text{Re } K^{(0)} + \mathcal{G}^{-1} \nabla \text{Im } K^{(0)}, \quad (57)$$

$$\dot{B} = -BK_{yy}^{(0)}B - BK_{yx}^{(0)} - K_{xy}^{(0)}B - K_{xx}^{(0)}, \quad (58)$$

$$\dot{\alpha} = \frac{i\hbar}{4} \text{Tr}(\dot{B}B^{-1}) + Y \cdot \dot{X} - K(X, Y) + \frac{i\hbar}{2} \text{Tr}(K_{xy} + K_{yy}B), \quad (59)$$

where $Z = (X, Y)$, $\nabla := (\partial_x, \partial_y)$ is the double phase-space gradient, Ω_2 is the double phase-space symplectic form, and $K^{(0)}$ is the leading order term in the semiclassical expansion $K = K^{(0)} + \hbar K^{(1)} + \dots$. We have also defined

$$K_{xy} = \begin{pmatrix} \frac{\partial^2 K}{\partial_{x_q} \partial_{y_q}} & \frac{\partial^2 K}{\partial_{x_q} \partial_{y_p}} \\ \frac{\partial^2 K}{\partial_{x_p} \partial_{y_q}} & \frac{\partial^2 K}{\partial_{x_p} \partial_{y_p}} \end{pmatrix}, \quad (60)$$

$K_{yx} = (K_{xy})^T$ and \mathcal{G} is related to B via

$$\mathcal{G} = \begin{pmatrix} \text{Im } B & -\text{Re } B \text{ Im } B^{-1} \\ -\text{Im } B^{-1} \text{ Re } B & \text{Im } B + \text{Re } B \text{ Im } B^{-1} \text{ Re } B \end{pmatrix}. \quad (61)$$

In order to obtain some insight into the properties of these equations of motion we have to compute the real and imaginary parts of $K^{(0)}$,

$$\text{Re } K^{(0)} = H^{(+)} - H^{(-)} + \sum_k \text{Im} (\bar{L}_k^{(-)} L_k^{(+)}) , \quad (62)$$

$$\text{Im } K^{(0)} = -\frac{1}{2} \sum_k |L_k^{(+)} - L_k^{(-)}|^2 . \quad (63)$$

We see that the imaginary part is an even function of y , $\text{Im } K^{(0)}(x, -y) = \text{Im } K^{(0)}(x, y)$, with $\text{Im } K^{(0)}(x, 0) = 0$, and non-positive. The real part is an odd function of y , $\text{Re } K^{(0)}(x, -y) = -\text{Re } K^{(0)}(x, y)$. Using these properties and direct computations we find that

$$\nabla \text{Re } K^{(0)}(x, 0) = \left(0, \Omega \nabla_x H(x) + \Omega \sum_k \text{Im} (L_k(x) \nabla \bar{L}_k(x)) \right) , \quad (64)$$

$$\nabla \text{Im } K^{(0)}(x, 0) = (0, 0). \quad (65)$$

By inserting this result into (57) we observe that if the initial value of Y is 0 then Y stays 0 for all times, and X satisfies the same equation of motion we found in (15). If we insert $B = 2iG$ into (58), and separate the real and imaginary parts of the Hessian matrix of $K^{(0)}$ at $y = 0$ in the same way, we find similarly that G satisfies (16). Hence our two different approaches are consistent.

The next natural question to ask is what happens when $Y \neq 0$. In this case we have a highly oscillatory initial Wigner function, which corresponds to a very non-classical state. As the imaginary part $\text{Im } K^{(0)}(x, y)$ in (63) has a maximum at $y = 0$, we see that the gradient part in (57) wants to push Y to 0, and hence reduce the frequency of the oscillations. In addition, (59) generates an exponentially damping factor if $\text{Im } K^{(0)}(x, y) < 0$, and both these effects are directly induced by the Lindblad operators. Thus, the general structure of the equations of motion (57) and (59), together with (63), allow us to conclude that oscillatory initial conditions will be smoothed and suppressed exponentially fast if they couple to the Lindblad terms via (63). This is a manifestation of decoherence [28] and we will illustrate this in the next example.

Example 4: We now show how this approach allows for a treatment of interference terms in the Wigner function by obtaining the semiclassical dynamics of an initial Schrödinger cat state in a damped anharmonic oscillator. Working in units with $\hbar = 1$, the Hamiltonian is

$$\hat{H} = \frac{1}{2}(\hat{q}^2 + \hat{p}^2) + \frac{\beta}{4}\hat{q}^4, \quad (66)$$

where β controls the degree of anharmonicity, and the damping is modelled with the Lindblad operator

$$\hat{L} = \sqrt{\frac{\gamma}{2}}(\hat{q} + i\hat{p}), \quad (67)$$

where γ determines the damping rate. When $\beta = 0$ the semiclassical dynamics are exact. The corresponding double phase-space symbol K is found from (55) to be

$$K(x, y) = (\Omega x) \cdot y - \frac{\beta}{4}(x_q y_p^3 + 4x_q^3 y_p) - \frac{\gamma}{2}x \cdot y - \frac{i\gamma}{4}(y \cdot y - 2). \quad (68)$$

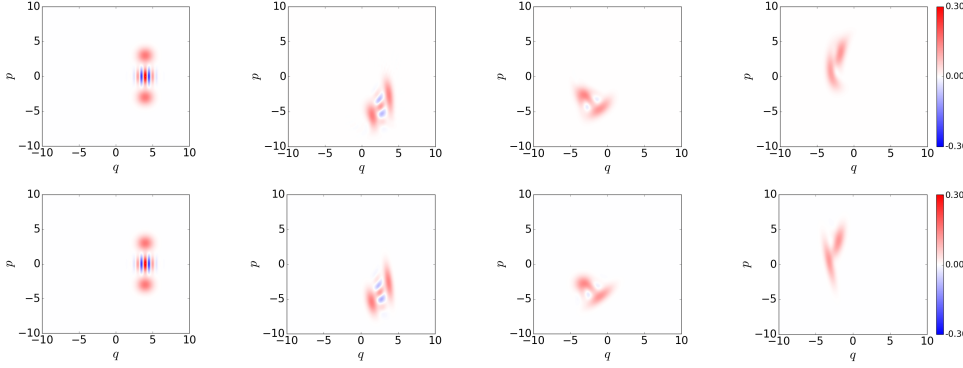


Figure 7: The quantum (top) and semiclassical (bottom) dynamics of an initial cat state in an anharmonic potential with $\beta = 0.1$ and damping at a rate $\gamma = 0.3$. Times $t = 0, 0.5, 1.5, 2.5$ are shown from left to right.

We consider an initial state that is a cat state, comprised of two Gaussians with $A = 1$ and both centred at $q = 4$ with momenta $p = \pm 3$. The Wigner function of this state (depicted on the left of Figure 7) is composed of two Gaussians centred at $(q = 4, p = \pm 3)$ in phase-space, and an interference pattern centred at the midpoint of the two Gaussians with a Gaussian envelope. The semiclassical dynamics are obtained by evolving each complex Gaussian component of the initial Wigner function, summing up the results, and applying the Wigner function normalisation condition $\int dx W(x) = 1$. The resulting semiclassical dynamics are compared with the quantum dynamics in Figure 7. The Gaussian approximation clearly reproduces the essential features of the oscillation, damping and decoherence.

In Figure 8 we further depict a comparison between the semiclassical position and momentum expectation values and the quantum results for a longer timescale. We find that there is good agreement at short times, while at longer times the results deviate as expected, while still capturing the qualitative features of the dynamics.

5. Summary

We have investigated the dynamics of Gaussian states in open quantum systems described by Lindblad equations in the semiclassical limit. This yields a new type of classical phase-space dynamics incorporating the effects of damping and decoherence in the dynamics of the phase-space coordinates as well as a semiclassical approximation for the quantum covariances. This dynamics has an interesting geometric structure,

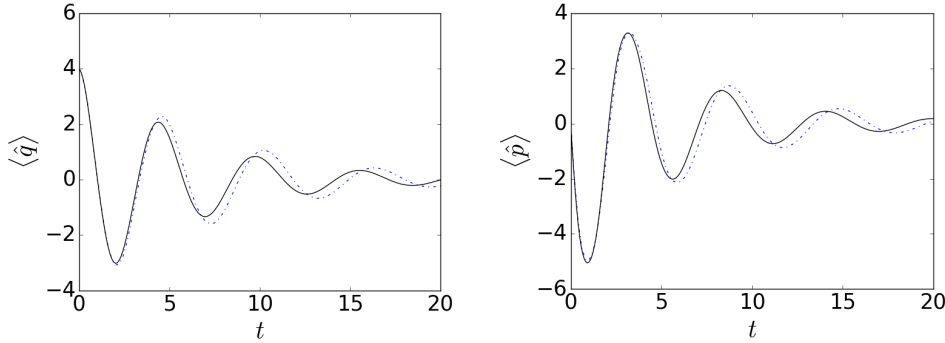


Figure 8: Time evolution of the position (left) and momentum (right) expectation values of the cat state above. The quantum dynamics (black) are compared to the semiclassical results (blue dashed). Anharmonic parameter $\beta = 0.1$ and damping rate $\gamma = 0.3$.

which we have explored in a number of example systems. We have also transformed the dynamics to complex phase-space variables, as they appear naturally in many models in quantum optics.

What makes the dynamics of Gaussian wave packets particularly appealing for closed quantum systems is the fact that an arbitrary initial wave function can be expanded into Gaussian states, each of which can be propagated independently. This expansion can also be repeated at intermediate time-steps, allowing for numerical quantum dynamics that can in principle be arbitrarily accurate [7, 8]. Due to the fact that Lindblad dynamics generates mixed states, this approach cannot be directly generalised to the case of open systems considered here. However, we have demonstrated how the interpretation of the Wigner dynamics in phase-space as a Schrödinger dynamics with non-Hermitian Hamiltonian can be used to circumvent this issue. We have demonstrated that this allows for the semiclassical propagation of interference terms by considering a cat state in a damped anharmonic oscillator.

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